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# Reflective Plausible Reasoning in Solving Inequality Problem

Imam Rofiki[1], Toto Nusantara[2], Subanji[2], Tjang Daniel Chandra[2]
[1]*Doctoral student in Mathematics Education, State University of Malang, Indonesia*
[2]*Department of Mathematics, State University of Malang, Indonesia*

***Abstract:*** *This study explored students' reflective plausible reasoning in solving inequality problem. This explorative study with the qualitative approach was conducted to seven subjects. Data are derived from the result of written answer, think aloud, and interview. The data from those subjects were analyzed using a constant comparative method so that it was obtained the same characteristics of reflective plausible reasoning. In this article, the authors described two subjects. The results of this study were the characteristics of students' reflective plausible reasoning shown by these behaviors: (1) students gave the argumentations based on intrinsic mathematical properties during solving inequality problem, (2) students experienced state of perplexity in problem solving process, (3) students realized that there was inaccuracy in the problem solving process which is indicated by feeling suspicious, doubtful, or curious, (4) students conducted inquiry to correct their error until they found the right solution, and (5) students experienced state of steadiness which is indicated by feeling sure and satisfied toward the truth of the result.*

***Keywords:*** *Reflective plausible reasoning, problem solving, inequality, intrinsic mathematical properties*

## I. Introduction

Reasoning and problem solving are two components which are close interrelated. The researchers and psychologist have tried to get the students' reasoning process by analyzing their argumentation during problem solving. Chi, Bassok, Lewis, Reimann, and Glaser [1] examine the students' argumentation in problem solving as the way to get deep knowledge that is being the basis of success in problem solving. Chi et al. conclude that successful problem solver is the one who can make the inference from the given information and give the explanation about the activity done in problem solving.

Mathematical reasoning is one of a basic mathematics competence that is essential to be trained to the students. Basic mathematics competence includes problem solving ability, reasoning ability, and conceptual understanding [2]. Mathematical reasoning is vital to be used in understanding mathematics. By the mathematical reasoning, mathematics can be understood by student meaningfully [3]. Mathematical reasoning is very important for mathematics education research. Kamol and Har [4] reveal the importance of knowing the way of students' thinking and reasoning to increase the students' learning achievement in mathematics, especially the success in mathematical problem solving. Peretz [5] emphasizes that students need to reason and develop the reasoning on their mind.

Polya [6] divides reasoning into two kinds, namely demonstrative reasoning and plausible reasoning. In plausible reasoning, the main thing is to differentiate a more reasonable guess from a less reasonable guess, whereas in demonstrative reasoning the main thing is to differentiate a proof from a guess, that is the demonstration of a valid proving from the effort of an invalid proving. Furthermore, Polya explains that people assure their knowledge by demonstrative reasoning, but they support their conjecture by plausible reasoning. Polya views the inductive reasoning as the certain case of plausible reasoning. The demonstrative reasoning is also called as strict reasoning [6] or proof reasoning [7].

By referring to the Polya's idea about plausible reasoning but it is not the definition, Lithner [7] characterizes the reasoning process of university students in solving mathematical task into two kinds, namely plausible reasoning (PR), and reasoning based on established experience. Furthermore, the latter term is abbreviated by EE. PR and EE are the extension of analytical thinking process and pseudo-analytical thinking process proposed by Vinner [8]. The analytical thinking process happens when a person faces a structure of a complex problem and his/her scheme does not reach it, so the person will solve the problem into simpler parts that can be reached out. The difference between analytical thinking process and PR is the degree of certainty in reasoning. The degree of certainty in PR is higher than analytical thinking process. Meanwhile, the difference between pseudo-analytical thinking process and EE is on the degree of analyticity. The pseudo-analytical thinking process is not analytical thinking process, but EE has analytical thinking content, though only a few. Students who apply pseudo-analytical thinking process can produce a wrong solution or a right solution.

Lithner [7] defines PR in mathematical task solving if the argumentation: (i) is based on mathematical properties of the component involved in the reasoning, and (ii) is meant to guide toward the truth without necessarily having to be complete and correct. This component is related to the fact, concept, definition, operation, principle (axiom, property, theorem, lemma, or corollary), action, process, object, procedure, or heuristic. Lithner explains that plausible reasoning is an extended and a looser version of proof reasoning, but it is still based on the mathematical property. This mathematical property refers to intrinsic mathematical property. The intrinsic mathematical property is a property that is relevant to mathematical task solving. It is accepted by mathematical society as correct. The opponent of the intrinsic mathematical property is surface property. The surface property has no (or a little) relevance to task solving. Plausible reasoning includes proof as a special case with the difference that proof requires to a higher degree of certainty in the formal mathematical proof, such as: complete, correct, and based on deductive logic. Reasoning will be called EE if the argumentation: (i) is based on the ideas and procedures built on the one's previous experience from the learning environment, and (ii) is meant to guide toward the truth without necessarily having to be complete and correct [7]. Condition (ii) in the definition of EE is same as the condition (ii) in the definition of PR because the purpose of reasoning is same. The fundamental difference between the definition of PR and EE is on the argumentation as described by the condition (i). In EE, the argumentation is commonly the transfer of property from one situation of familiar task solving to another situation that has some similarity. Reasoning done by students in EE is often superficial, without considering intrinsic mathematical property from the component involved in the reasoning. Students use the procedure of task solving only based on their previous experience without understanding. In this study, PR is defined as reasoning by giving argumentation based on intrinsic mathematical properties. Whereas EE is reasoning by giving argumentation based on idea and procedure constructed from the previous experience without deep understanding.

The studies about PR and EE in solving mathematical task have been examined by some researchers [7], [9], [10], [11]. Cawley [9] finds that many university students use EE in solving the task of linear equation. Rofiki et al. [11] find that university student gets the right answer in the problem solving of the quadratic equation but the university student cannot give argumentation based on intrinsic mathematical properties. The university student only transfers the old knowledge to solve the problem without deep understanding. Hence, the university student applies EE. Meanwhile, Lithner shows that many university students do EE and they get difficulty in doing PR [7], [10].

Students maybe have a difficulty in problem solving so that they do reflective thinking process. Dewey [12] defines that reflective thinking as active, continue, and careful thinking which supported a conviction/knowledge and an invention of problem solution. John Dewey is the first expert who introduces the idea of reflective thinking process in education. Furthermore, Dewey explains that reflective thinking process moves from a perplexity state (also being called as disequilibrium) as unclear situation, doubtfulness, conflict, and disorder thinking to a clear situation, coherence, harmony, and steady state (equilibrium). Perplexity happens when the student faces a problem situation that the complete solution scheme has not been known clearly. The student's internal experience has not been wholly used maximally. This condition became one cause of disequilibrium and unsteadiness thinking. This will awake student's intention to balance his/her thinking process so that it will encourage the student to solve the problem, i.e. to start the inquiry process. Hence, it can be concluded that reflective thinking process is thought process happened when a student experiences perplexity and do the inquiry to find the solution of the problem. By referring to the definition of PR and Dewey's definition of reflective thinking, reflective plausible reasoning in this study is defined as PR followed by reflective thinking process in problem solving.

One of the ways that can be used to explore students' plausible reasoning is using problem solving. The students are asked to solve inequality problem (non-routine task). The inequality problem in this study is the question of inequality that can be understood by students and it is challenging for them but it cannot be solved by a routine procedure that known by them. To gain the right procedure, it is needed a deep thinking and analysis. In other words, students have the aim to solve the problem but the complete solution scheme is not available immediately on their mind.

Inequality, particularly the solution set of inequality is an essential concept in calculus because the main discourse of calculus involves function concept and analysis of the function property. The analysis of the property of particular function needs the solution set of inequality such as in determining the domain of the irrational function. Moreover, the solution set of inequality is needed to find the monotonicity and concavity of functions by applying the derivative concept. Consequently, students need to understand the inequality concept well in order to gain success in learning calculus. Students also need to learn inequality concept to build comprehension in trigonometry, geometry, discrete mathematics, linear programming, algebra, and real analysis.

The inequality concept becomes an interesting topic to be studied. Because of the importance of inequality concept in calculus and the other fields of mathematics, this causes increasing studies about inequality. Yet, some studies show that students get difficulty in solving the inequality problem. Bazzini and Tsamir [13] find that many students have some problems to solve algebraic inequality. Meanwhile, Fujii [14] finds that students experience difficulty in finding the solution set of inequality which yield real number set. Sierpinska [15] finds the candidate of mathematics and statistics university students who does not realize their error in determining the solution set of absolute value inequalities. Students do not know whether their answer is correct or wrong. Students depend on lecturer's argumentation about the truth of their answer. Students can solve the problems that have the similar steps with the example from their lecturer but to solve the other problems that need PR, the students cannot give argumentations based on intrinsic mathematical properties.

The purpose of this study is to explore students' reflective plausible reasoning in solving inequality problem. Educators can use the result of this study as consideration for designing the learning strategies to increase the students' reflective plausible reasoning in the mathematics classroom. In addition, the result also gives the contribution to researchers as the theoretical framework or empirical facts about reflective plausible reasoning and inequality problem.

## II. Reasoning Structure And Characterization

Lithner uses the term of reasoning to all kinds of reasoning that related to mathematical task solving [7], [10]. The mathematical task can be an exercise (routine tasks) and a problem (non-routine tasks). Furthermore, Lithner defines reasoning as the line of thinking or the way of thinking that is used to produce statements and reach a conclusion in task solving. Related to reasoning, argumentation and justification solution are essential to reinforce or refuse a statement. Justification refers to the act of defending or clarifying statements [16]. While the argumentation is confirmation (verification), part of the reasoning aimed to convince oneself or others that the performed reasoning is correct [7], [10].

To solve a mathematical task, students can solve a set of subtasks. The way that can be used to describe the reasoning in solving mathematical task is by structuring student's reasoning through 4 steps, namely 1) A problematic situation, 2) strategy choice, 3) strategy implementation, and 4) conclusion [7], [10]. This reasoning structure describes the line of student's reasoning in solving mathematical task starting from face the task to conclude the obtained result.

Lithner [10] proposed the modification of reasoning characterization by introducing the term of local plausible reasoning (LPR) and global plausible reasoning (GPR). Reasoning in mathematical task solving is called LPR if it satisfies at least one of the following two conditions: (i) the strategy choice is based on identifying similar surface properties in the task and component of situations in the text, but PR is used locally to determine whether the procedure can be copied to solve the task or not, or (ii) The strategy implementation is mostly based on copying the solution procedure from the identified situation, but one or a few local steps of this procedure are modified by construction of PR [10]. While reasoning in mathematical task solving is called GPR if it satisfies at least one of the following two conditions: (i) the strategy choice is mostly based on analysis and consideration of intrinsic mathematical properties from the components in the task. The idea is constructed and supported by PR, or (ii) the strategy implementation is mostly supported by PR [10]. The similarity between LPR and GPR is in the existence of PR whereas the difference is the range of PR. GPR concerns the whole solution by the implementation of PR globally while LPR applies PR locally. If a mathematical task is impossible to be solved by EE or LPR, then GPR or PR needs to be applied.

Based on the explanation above, LPR (GPR) is defined as PR applied locally (globally) in the whole of problem solving. In LPR, the student gives argumentation only in a few local parts by considering the intrinsic mathematical properties. While in GPR, argumentation is mainly based on considering the intrinsic mathematical properties.

The authors further make the position of Lithner's reasoning characterization based on the lens of the range of argumentation based on intrinsic mathematical properties (the lens of PR) in the of problem solving. The reasoning characterization includes EE, LPR, GPR, and PR. The position of the reasoning characterization is not discrete, but it is continuous. The authors relate this position with fuzzy theory. In the universe of a crisp set (a classical set), a membership function for a set (also called characteristic function, indicator function, or discrimination function) is expressed explicitly with 0 (if it is the element of a set) and 1 (if it is not an element of a set). Whereas fuzzy set allows the membership function to all values in the interval $[0,1]$. In other words, a membership function of a crisp set only has exactly two values (0 and 1) while membership function of the fuzzy set is a continuous function with range $[0,1]$. EE is not PR so the value of EE's membership function (also called membership degree) is 0 while the value of PR's membership function is 1. The membership degree of LPR and GPR is $\left(0,\frac{1}{2}\right)$ and $\left(\frac{1}{2},1\right)$, respectively. EE (PR) is shown in the leftmost position (the rightmost position). The membership degree refers to the whole of PR. The membership function moves increasingly from the leftmost to the rightmost. The position of EE, LPR, GPR, and PR is shown in Figure 1.

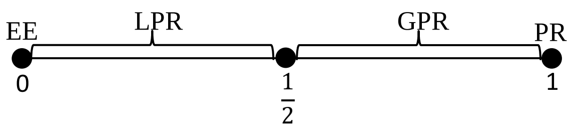



**Figure 1.** The position of EE, LPR, GPR, and PR

Students need to do the process of reflective thinking when they find imprecision in the process of problem solving. The reflective thinking is very crucial for the students because it is a rearrangement of thinking in order to understand and solve the problem. The role of the reflective thinking is that making students believe (or do not believe) their solution. If students do the reflective thinking until correcting the mistakes or finding the solution, then they will feel sure on their solution. On the contrary, students will not believe their solution if they have done the reflective thinking but they are not able to find the solution. The cause of students' failure in finding the solution of the problem is not optimally the process of their reflective thinking.

Relating to the reasoning characterization previously, the authors make the reasoning characterization with the lens of the existence of reflective thinking process. EE (LPR, GPR, or PR) followed by a process of reflective thinking in problem solving is called by a reflective EE (a reflective LPR, a reflective GPR, or a reflective PR), whereas EE (LPR, GPR, or PR) that is not followed by a process of reflective thinking in problem solving is called by a non-reflective EE (a non-reflective LPR, a non-reflective GPR, or a non-reflective PR). A reflective EE (a reflective LPR, a reflective GPR, a reflective PR, a non-reflective LPR, a non-reflective GPR, or a non-reflective PR) is abbreviated as $R_f$EE ($R_f$LPR, $R_f$GPR, $R_f$PR, $NR_f$EE, $NR_f$LPR, $NR_f$GPR, or $NR_f$PR). The position of reasoning characterization ($R_f$EE, $R_f$LPR, $R_f$GPR, $R_f$PR, $NR_f$EE, $NR_f$LPR, $NR_f$GPR, and $NR_f$PR) is presented in Figure 2.

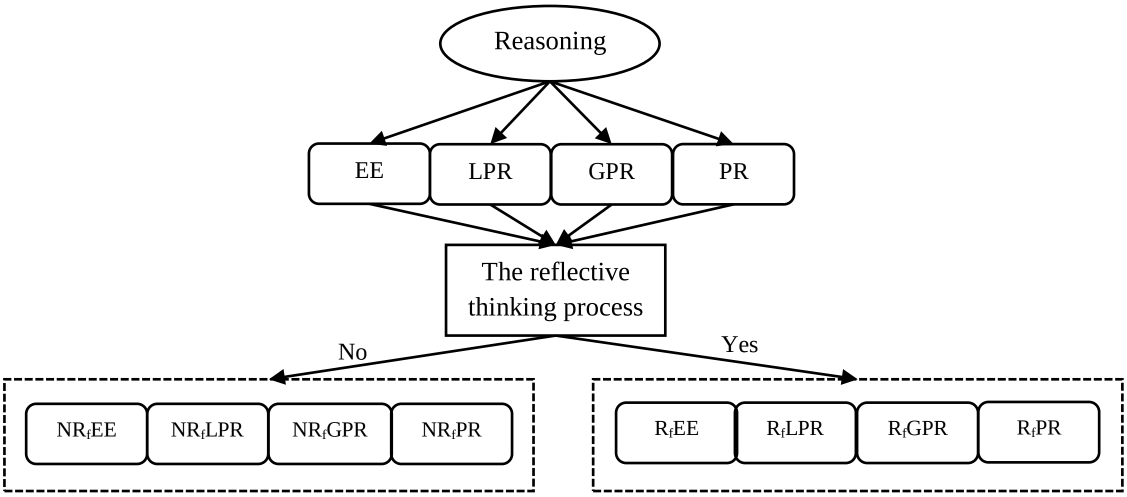



**Figure 2.** The position of reasoning characterization

## III. Method

This type of study was exploratory study using the qualitative approach. The number of undergraduate students included in this study was 41. The students are from a university located in East Java, Indonesia. The students who become the candidate of research subjects are not randomly selected, but they are chosen by 2 criteria, namely intending to be a subject and getting a recommendation from their lecturer. Furthermore, students who do $R_f$PR are selected as research subjects while students who do $R_f$LPR, $R_f$GPR, $R_f$PR, $NR_f$EE, $NR_f$LPR, $NR_f$GPR, or $NR_f$PR are not chosen as research subjects. Subject selection is done continuously until obtaining a saturation of data. The saturation of data means that the subject to each group has the same pattern. The data were analyzed with the constant comparative method. The method is called the constant comparative method because the analysis of the data in this study compares the data with the other data constantly, and then it compares the category with the other categories regularly [17], [18].

The data collection was carried out by giving a task of inequality problem solving to the subjects. The problem is to determine the set of all real numbers $x$ that satisfies the inequality $\sqrt{x^2-1} < 2x+2$. The subjects were asked to express aloud any words what their thinking at first receiving a problem to solving the problem. The authors recorded the subjects' utterance and the subjects' behavior, including the unique things done by the subjects when solving the problem. This data collection is called *think aloud* [3], [7], [10] or *think out loud/TOL* [19]. The think aloud method can be used to explore the process of students' cognition/thinking that can not be observed when students solve a problem [3].

The authors also interviewed the subjects to get information about data of $R_f$PR that has not been revealed in the written data and the think aloud. In addition, this interview was conducted for confirming the subjects' work. In the interview process, the subjects were asked to justify and explain what has been done and give reasons why they do or answer like that. The authors also recorded the subjects' conversation and the subject's behavior during the interview. After collecting the data, the authors transcribed the recording of the think aloud and the interview. Afterward, the authors analyzed the data from the result of the written answer, the think aloud, and the interview to get the characteristics of $R_f$PR.

## IV. Results And Discussion

Of the 41 students in this study, 19 students did EE (15 $NR_fEE$ and 4 $R_fEE$), 10 students did LPR (7 $NR_fLPR$ and 3 $R_fLPR$), 4 students did GPR (2 $NR_fPR$ and 2 $R_fGPR$), and 8 students did PR (1 $NR_fPR$ and 7 $R_fPR$). The following Table 1 shows the distribution of students' reasoning in solving inequality problem.

**Table 1.** The distribution of students' reasoning in solving inequality problem

| Reasoning | | | | | | | |
|---|---|---|---|---|---|---|---|
| EE | | LPR | | GPR | | PR | |
| 19 | | 10 | | 4 | | 8 | |
| $NR_fEE$ | $R_fEE$ | $NR_fLPR$ | $R_fLPR$ | $NR_fGPR$ | $R_fGPR$ | $NR_fPR$ | $R_fPR$ |
| 15 | 4 | 7 | 3 | 2 | 2 | 1 | 7 |

After the authors analyzed data in the $R_fPR$ group by a constant comparative method, it is obtained the result that seven subjects had the same characteristic of $R_fPR$. In this article, the authors described two subjects that are S1 (a male) and S2 (a female). According to the result of written answer, the think aloud, and interview transcript, the first activity done by the subjects was reading the problem many times. S1 read twice, whereas S2 read three times. Their reason behind that activity is to more accurate in understanding the information of problems such as the universe set of real number, inequality objects, and the problem question. A problematic situation met by the subjects appeared when they thought what should be done to determine the solution set. They thought hard indicated by silencing for a long time, holding the head, or asking the solution. After thinking hard, they arranged a strategy. In the strategy choice step, subjects described the problem at 3 cases. S1 and S2 explained $x^2 - 1 \geq 0$ as the first case. S1 explained the second and the third case as $\sqrt{x^2 - 1} < 2x + 2$ and $2x + 2 > 0$, respectively. Whereas S2 explained the second and the third case as $2x + 2 > 0$ and $\sqrt{x^2 - 1} < 2x + 2$, respectively.

In the strategy implementation step, S1 and S2 determined the property of root value as the first case (the first requirement), namely $x^2 - 1 \geq 0$. The subjects gave argumentation that the radicand has to greater than or equal to zero in order to the result value is still the element of the real number set. If the radicand is less than zero, then the result value is an imaginary number. The imaginary number is not an element of a real number set. On the other hand, it is an element of a complex number set. S1 factorized $x^2 - 1 \geq 0$ into $(x + 1x - 1 \geq 0$. S1 showed equivalent of $x+1x-1 \geq 0$ to $x+1x-1>0$ or $x+1x-1=0$. Further, S1 used theorem in real number system such as 1) if $ab > 0$ then ($a > 0$ and $b > 0$) or ($a < 0$ and $b < 0$), and 2) if $ab = 0$ then $a = 0$ or $b = 0$. S1 analyzed all solutions possibilities by applying set and logic concepts to take a decision in determining the solution set. Moreover, S1 used inequality concept, namely adding/subtracting the same quantity to both sides of inequality will get the equivalent inequality with the previous inequality. The solution set obtained by S1 was $\{x | x \leq -1 \; or \; x \geq 1, x \in \mathbb{R}\}$. The S1's written answer in the first case is shown in Figure 3.

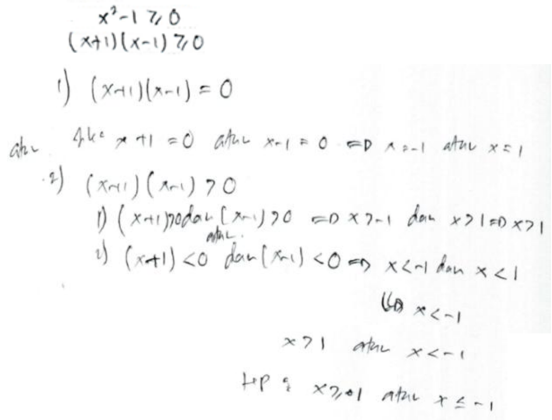


**Figure 3.** The S1's written answer in the first case $(x^2 - 1 \geq 0)$

While S2 added 1 to both sides $x^2 - 1 \geq 0$ so it is obtained $x^2 \geq 1$. S2 took square root on both sides of $x^2 \geq 1$ so it is obtained $x \geq 1$. After getting this result, S2 was silent for a long time while moving the forefinger. S2 experienced perplexity and asked the truth of the obtained result. S2 said slowly that "*is my answer correct? I think there is a problem in my way.*" S2 was doubtful and suspicious with her problem solving strategy. By this suspicious, S2 did inquiry all solving steps that have to be done. After thinking hard. S2 realized that her answer was wrong. S2 expressed that there were 2 possibilities of x real number that satisfy $x^2 \geq 1$, namely $x \leq -1$or $x \geq 1$. Her reason was the squaring for every real number in $x \leq -1$ or $x \geq 1$ is greater than or equal to 1. S2 tried to use another strategy for convincing the truth of her solution. S1 factorized $x^2 - 1 \geq 0$ so that it is obtained $(x+1)(x-1) \geq 0$. S2 used the property that if $ab \geq 0$ then $(a \geq 0$ and $b \geq 0)$ or $(a \leq 0$ *and* $b \leq 0)$. S2 got 2 possibilities, namely $\big((x+1) \leq 0$ *and* $(x-1) \leq 0\big)$ or $\big((x+1) \geq 0$ and $(x-1) \geq 0\big)$. By applying set, logic, and inequality concepts, S2 obtained $\{x | x \leq -1$ or $x \geq 1, x \in \mathbb{R}\}$ as the solution set of $x^2 - 1 \geq 0$. S2 was sure and satisfied with the truth of her answer because she has gotten the same result by two different strategies. The thought process done by S2 showed that the characteristics of a reflective thinking process. The S2's written answer in the first case is shown in Figure 4.

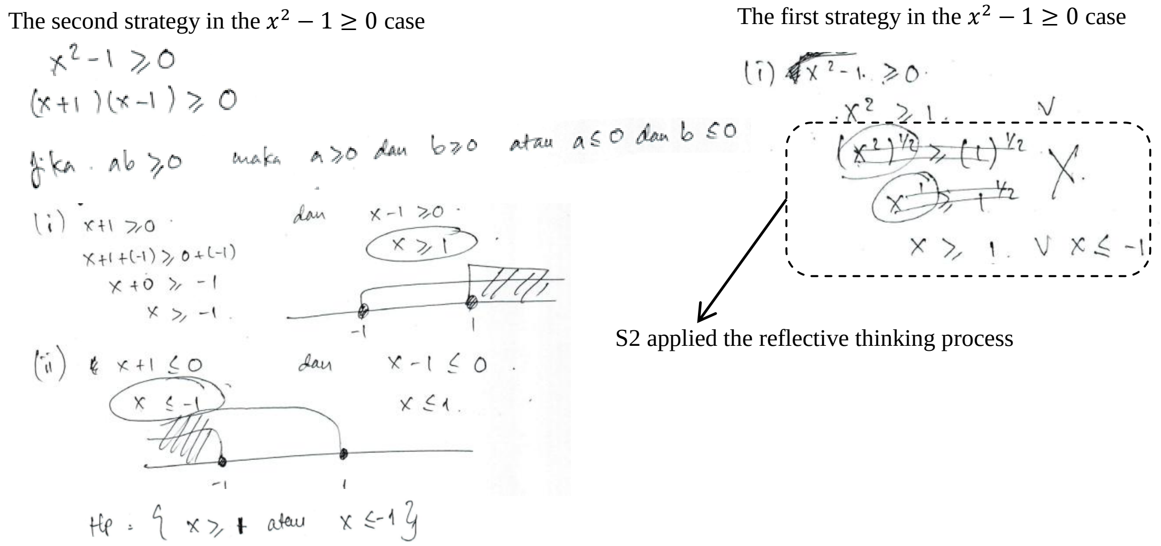



**Figure 4.** The S2's written answer in the first case $(x^2 - 1 \geq 0)$

S2 determined $2x + 2 > 0$ as the second case. S2 analyzed 3 possibilities of value in $2x + 2$, namely positive real number, negative real number, or zero. S2 considered that the left side value of $\sqrt{x^2 - 1} < 2x + 2$ was non-negative real number. Furthermore, S2 got the result of her analysis, namely 1) if the left side is non-negative and the right side is negative, then it does not satisfy the inequality problem because non-negative is not less than negative, 2) if the left side is non-negative and the right side is zero, then it does not satisfy the inequality problem because non-negative is not less than zero, or 3) if the left side is non-negative and the right side is positive, then it satisfies the inequality problem. S2 gave the reason about the third condition, namely because the minimum value on the left side of $\sqrt{x^2 - 1} < 2x + 2$ is zero so the right side is greater than zero. Further, S2 subtracted 2 to both sides of $2x + 2 > 0$ and multiplied $\frac{1}{2}$ to both sides of the obtained inequality so that S2 got the solution set of the second case, that is $\{x | x > -1, x \in \mathbb{R}\}$. The S2's written answer in the second case is shown in Figure 5.

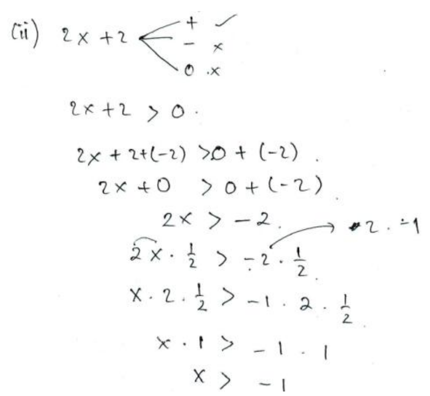


**Figure 5.** The S2's written answer in the second case $(x^2 - 1 \geq 0)$

S2 determined $\sqrt{x^2 - 1} < 2x + 2$ as the third case, whereas S1 determined it as the second case. The subjects squared both sides of $\sqrt{x^2 - 1} < 2x + 2$ so that it is obtained $x^2 - 1 < 4x^2 + 8x + 4$. They justified that squaring both sides of the inequality can be done because the value of left side and the right side is non-negative and positive, respectively. They showed that it is can not be done if one of both sides inequality is negative. A counterexample given by S1 is $-2 < 1$ but $(-2)^2 = 4 \nless 1^2 = 1$. Whereas S2 gave a counter example that $-3 < -1$ but $(-3)^2 = 9 \nless (-1)^2 = 1$. The subjects subtracted $x^2$ and added 1 to both sides so that it is obtained $0 < 3x^2 + 8x + 5$. That result is equivalent to $3x^2 + 8x + 5 > 0$. They factorized $3x^2 + 8x + 5 > 0$ into $(3x+5)(x+1) > 0$. They used the property if $ab > 0$ so $(a > 0 \text{ and } b > 0)$ or $(a < 0 \text{ and } b < 0)$. They got 2 possibilities, namely $\big((3x+5) > 0 \text{ and } (x+1) > 0\big)$ or $\big((3x+5) < 0 \text{ and } (x+1) < 0\big)$. They also used the concept of inequality, set, and logic to take the decision in determining the solution set. They obtained the solution set of $\sqrt{x^2-1} < 2x+2$, namely $\left\{x | x < \frac{-5}{3} \text{ or } x > -1, x \in \mathbb{R}\right\}$. The S2's and S1's written answer in the $\sqrt{x^2-1} < 2x+2$ case are shown in Figure 6 and Figure 7, respectively.

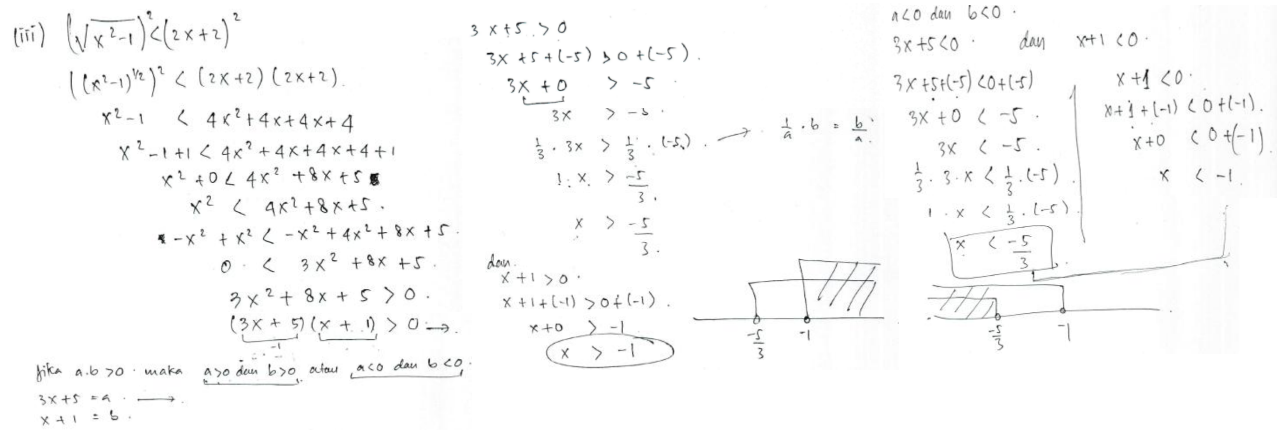


**Figure 6.** The S2's written answer in the third case $\left(\sqrt{x^2-1} < 2x+2\right)$

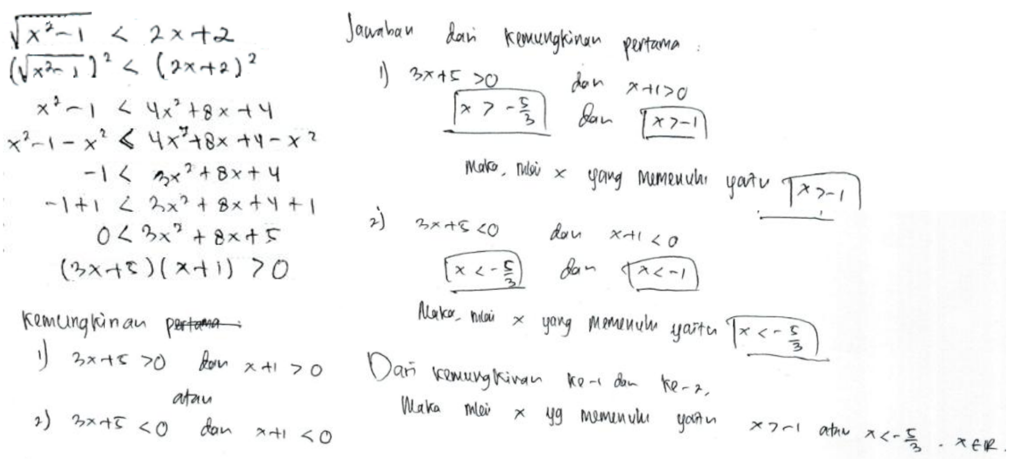


**Figure 7.** The S1's written answer in the second case $\left(\sqrt{x^2-1} < 2x+2\right)$

To determine the solution set of inequality problem, S2 intersected the solution set of the first, the second, and the third case. S2 got $\{x | x \geq 1, x \in \mathbb{R}\}$. The S2's written answer in determining the solution set is shown in Figure 8.

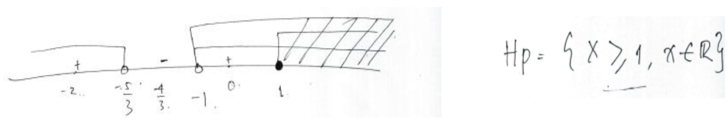


**Figure 8.** The S2's written answer in determining the solution set inequality problem

Whereas S1 intersected the solution set of the first and the second case to determine the solution set of inequality problem. S1 got $\left\{x | x < \frac{-5}{3} \text{ or } x \geq 1, x \in \mathbb{R}\right\}$. S1 checked the truth of the solution set by substituting some values of x ($x = 1$, $x = 2$, and $x = -2$) to inequality problem. S1 explained that $x = 1$ and $x = 2$ fulfilled $\sqrt{x^2-1} < 2x+2$ because $0 < 4$ and $\sqrt{3} < 6$ was correct statement. Whereas the result of substitution $x = -2$ did not fulfill $\sqrt{x^2-1} < 2x+2$ because $\sqrt{3} < -2$ is the wrong statement. Therefore, S1 was suspicious with the truth of $\left\{x | x < \frac{-5}{3}, x \in \mathbb{R}\right\}$. S1 experienced a complex perplexity. It seemed when S1 was silent for a long time while holding a head. S1 questioned the truth of $\left\{x | x < \frac{-5}{3}, x \in \mathbb{R}\right\}$ as the solution set of inequality problem. S1 was doubt and curiosity with his solution. S1 said, "*My answer maybe is wrong. How could it be? How do get the right solution?*" Because of those conditions, S1 did inquiry toward his problem solving previously. After thinking hard, S1 was sure that $\left\{x|x < \frac{-5}{3}, x \in \mathbb{R}\right\}$ did not satisfy the solution set of inequality problem. His thought process indicated that reflective thinking process occurs. S1 argued that for showing the statement is wrong it is sufficient to give one counterexample. His counterexample was $x = -2$. Hence, S1 realized that the obtained solution set was incorrect. S1 rechecked the problem solving in the first and the second case. S1 found the connection between the first and the second case, namely squaring process can be done when the left side and right side of the inequality is non-negative and of positive, respectively. His argumentation was because the minimum value on the left side of the inequality is 0 so $2x + 2$ has to greater than 0. Thus, S1 determined $2x + 2 > 0$ as the third case (also called by the second requirement). S1 asserted that the third case is crucial as the complement of two cases previously. Without involving the third case, the solution is not complete. By subtracting 2 to both sides of $2x + 2 > 0$ and multiplying $\frac{1}{2}$ to both sides of the obtained inequality, S1 found the solution set of the third case, namely $\{x|x > -1, x \in \mathbb{R}\}$. The S1's written answer in the third case is shown in Figure 9. Furthermore, S1 intersected the solution set of the first, the second, and the third case. S1 got $\{x|x \geq 1, x \in \mathbb{R}\}$. S1 wrote the solution set of inequality problem as shown in Figure 10.

**Figure 9.** The S1's written answer in the third case $(2x + 2 > 0)$

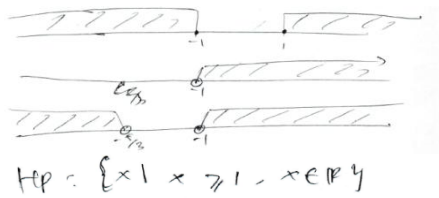


**Figure 10.** The S1's written answer after applying reflective thinking

In the conclusion step, subjects concluded that the solution set of $\sqrt{x^2 - 1} < 2x + 2$ was $\{x|x \geq 1, x \in$ *ℝ.* Subjects justified that the steps used to solve the problem were right because they have applied the mathematical properties and mathematical concepts. They really believed that the obtained result was correct. In convincing the result, they gave the general statement. They stated that $\sqrt{x^2 - 1} < x$ for every $x \geq 1$. Their reason was because $\sqrt{x^2}$ is equal to $x$ for every $x \geq 1$ and the value of $\sqrt{x^2 - 1}$ is less than $\sqrt{x^2}$ for every $x \geq 1$. They also gave argumentation that $x < 2x + 2$ for every $x \geq 1$ because $2x$ is greater than $x$ where $x$ is a positive real number so that $2x + 2$ is always greater than $x$ for every $x \geq 1$. They justified that $\sqrt{x^2 - 1}$ is less than $2x + 2$ for every real number in $x \geq 1$ because applying the transitive property in $\sqrt{x^2 - 1} < x$ and $x < 2x + 2$. It shows that they proved the validity of the result generally by including algebraic property, transitive property, and order property of real number set. They made the logical inference based on the transitive property. In proving the result, they could give a logical reason. According to Harel and Sowder [20], their proof scheme is classified by an analytic proof scheme. Whereas if it is viewed by Balaceff's proof taxonomy, their proof is a conceptual proof [21]. Figure 11 below shows their proving to convince the validity of the solution set.

The S1's written answer | The S2's written answer

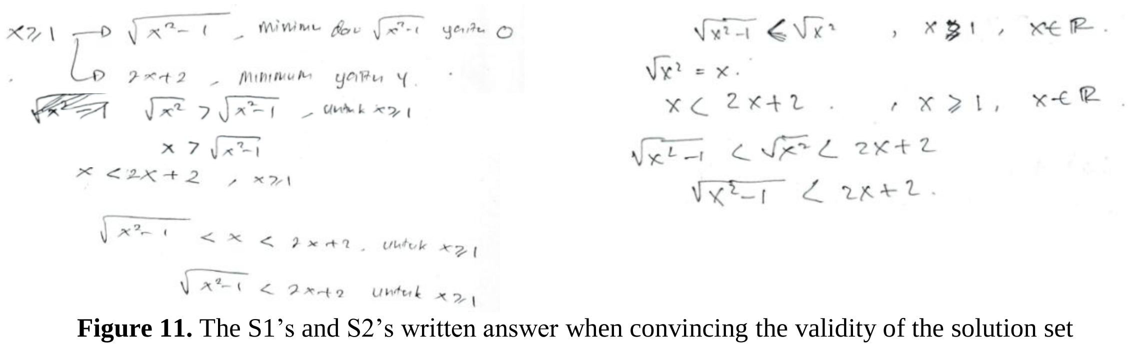


**Figure 11.** The S1's and S2's written answer when convincing the validity of the solution set

The subjects explained well the problem solving in each case. They gave argumentation based on intrinsic mathematical properties such as distributive property, inequality property, inequality concept, factoring concept, set concept, and logic concept. They experienced the perplexity in problem solving. They felt curiosity about their process of problem solving. They doubted the truth of their solution. This leads them to inquire the inaccuracy on their solution. After the main matter was founded, they realized that there was something wrong with it. Finally, they corrected it. They felt satisfied with the result. This indicates the condition of their steady thinking. According to Dewey [12], their mental process can be categorized as a reflective thinking process. It can happen because they are doubt or curiosity of what the problem truly is or how exactly a solution is. They performed reflective thinking process well because of their tenacity in finding the solution. Moreover, they have a deep knowledge of the material and a logical thinking ability.

Based on the S1's and S2's reasoning process in the discussion above, they experienced $R_f$PR. Their reasoning was shown by the giving argumentations based on mathematical intrinsic properties but they also applied reflective thinking process. The structure of their $R_f$PR is presented in Figure 12.

The structure of S1's $R_f$PR — The structure of S2's $R_f$PR

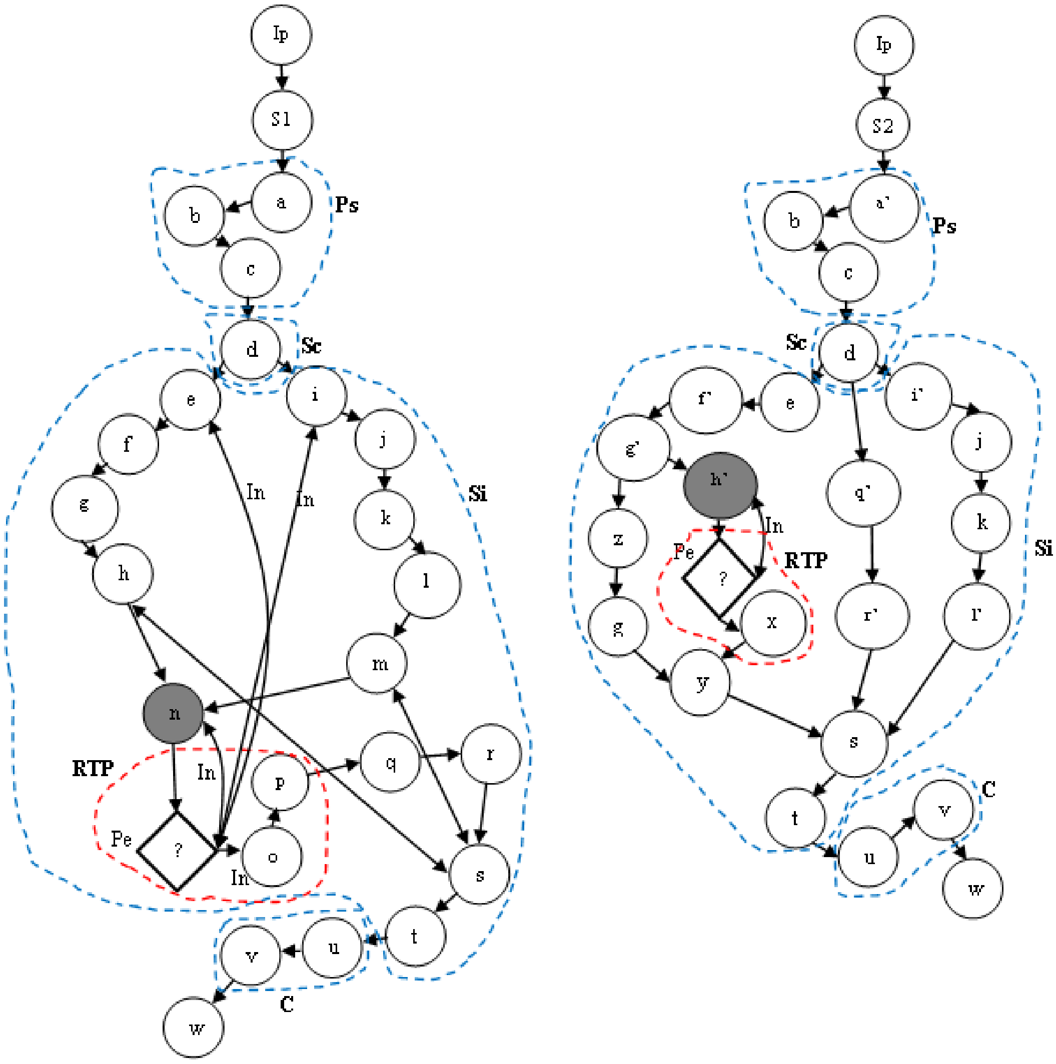



**Figure 12.** The structure of S1's and S2's $R_f$PR

Based on the analysis of subjects group, there were 5 same characteristics in $R_f$PR. The characteristics are (1) the existence of giving argumentations based on intrinsic mathematical properties during inequality problem solving, (2) the existence of a perplexity state in the problem solving process, (3) the existence of awareness about some inaccuracies in the problem solving process, (4) the existence of an inquiry to correct the error until finding the solution set of inequality problem, and (5) the existence of steady thinking followed by feeling sure and satisfied toward the truth of obtained result. In general, the process of $R_f$PR can be illustrated in Figure 13.

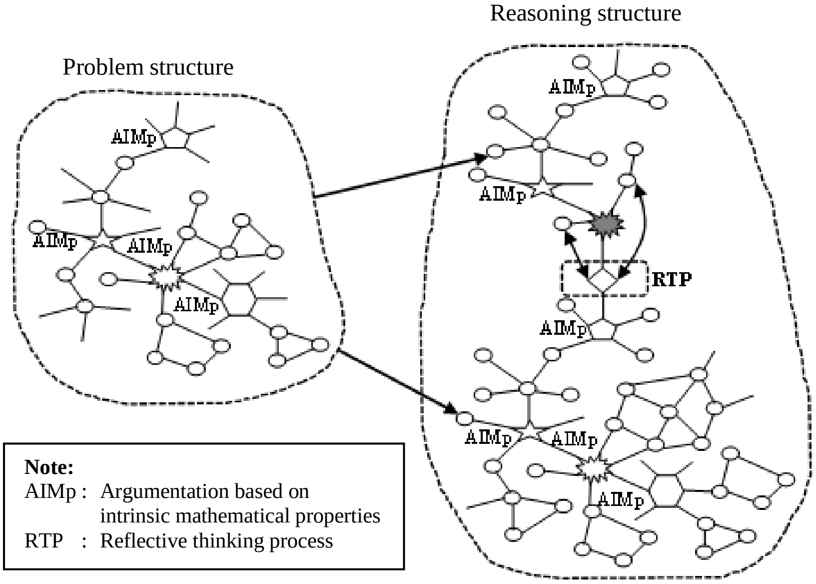
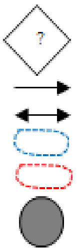



**Figure 13.** The process of reflective plausible reasoning ($R_fPR$)

**Code description of the reasoning structure in Figure 12:**



: Questioning the truth of the solution or the problem solving process
: The line of reasoning
: Two related activities
: The reasoning structure
: Reflective thinking process (RTP)
: Inaccuracy of solution

| | | |
|---|---|---|
| Ip | : | Inequality problem |
| S1 | : | The first subject |
| S2 | : | The second subject |
| a | : | Understanding inequality problem by reading two times |
| b | : | Mentioning the information in the problem |
| c | : | Thinking about what should be done to find the solution set of problem |
| d | : | Choosing a strategy by dividing the problem into several cases |
| e | : | Determining the 1st case, i.e. $x^2 - 1 \geq 0$ (because the universe set is in a set of real number) |
| f | : | Factorizing $x^2 - 1 \geq 0$ into $(x+1)(x-1) \geq 0$ |
| g | : | Determining the solution set of the 1st case using mathematical properties, inequality, set, and logic concepts |
| h | : | Gaining the solution set of the 1st case, i.e. $\{x \mid x \leq -1 \text{ or } x \geq 1, x \in \mathbb{R}\}$ |
| i | : | Determining the 2nd case, i.e. $\sqrt{x^2 - 1} < 2x + 2$ |
| j | : | Squaring both sides of $\sqrt{x^2 - 1} < 2x + 2$ (Because the left side is non-negative and the right side is positive) |
| k | : | Explaining the simplifying process from $\left(\sqrt{x^2 - 1}\right)^2 < (2x+2)^2$ until $(3x+5)(x+1) > 0$ |
| l | : | Determining the solution set of the 2nd case using mathematical properties, inequality, set, and logic concepts |
| m | : | Gaining the solution set of the 2nd case, i.e. $\left\{x \mid x < \frac{-5}{3} \text{ or } x > -1, x \in \mathbb{R}\right\}$ |
| n | : | Gaining the problem solution set by intersecting the result in h and m, i.e. $\left\{x \mid x < \frac{-5}{3} \text{ or } x \geq 1, x \in \mathbb{R}\right\}$ |
| o | : | Checking the truth of problem solution set by substituting $x = 1, x = 2$, and $x = -2$ to inequality problem |
| p | : | Realizing the inaccuracy in n because $x = -2$ did not satisfy inequality problem |
| q | : | Determining the 3rd case, i.e. $2x + 2 > 0$ (Because 0 is the minimum value of the left side inequality so that $2x + 2$ has to greater than 0) |
| r | : | Applying the concept of inequality in order to obtain the solution set of the 3rd case, i.e. $\{x \mid x > -1, x \in \mathbb{R}\}$ |
| s | : | Determining the problem solution set by intersecting the solution sets in the 1st, 2nd, and 3rd case |
| a' | : | Understanding inequality problem by reading three times |
| f' | : | Adding both sides of $x^2 - 1 \geq 0$ by 1 so that obtaining $x^2 \geq 1$ |
| g' | : | Taking square root on both sides of $x^2 \geq 1$ |
| h' | : | Gaining the solution set of 1st case, i.e. $\{x \mid x \geq 1, x \in \mathbb{R}\}$ |
| x | : | Realizing inaccuracy in the result because $x \leq -1$ also satisfies the solution of $x^2 \geq 1$ |
| y | : | Gaining the solution set of the 1st case, i.e. $\{x \mid x \leq -1 \text{ or } x \geq 1, x \in \mathbb{R}\}$ after applying RTP |
| z | : | Using the second way by factorizing $x^2 - 1 \geq 0$ into $(x+1)(x-1) \geq 0$ |
| q' | : | Determining the 2nd case, i.e. $2x + 2 > 0$ (Because 0 is the minimum value of the left side inequality so that $2x + 2$ has to greater than 0) |
| r' | : | Applying the concept of inequality in order to obtain the solution set of the 2nd case, i.e. $\{x \mid x > -1, x \in \mathbb{R}\}$ |
| i' | : | Determining the 3rd case, i.e. $\sqrt{x^2 - 1} < 2x + 2$ |
| l' | : | Determining the solution set of the 3rd case using mathematical properties, inequality, set, and logic concepts so that gaining $\left\{x \mid x < \frac{-5}{3} \text{ or } x > -1, x \in \mathbb{R}\right\}$ |
| t | : | Justifying the problem solving steps |
| u | : | Concluding the obtained result, i.e. $\{x \mid x \geq 1, x \in \mathbb{R}\}$ |
| v | : | Convincing the result obtained by applying the transitive property |
| w | : | Finish |
| Pe | : | Perplexity |
| In | : | Inquiry |
| Ps | : | Problematic situation |
| Sc | : | Strategy choice |
| Si | : | Strategy implementation |
| C | : | Conclusion |

Students who did $R_fPR$ could give logical reasons why the rules/procedures work or can be applied. Moreover, they also gave the counterexample if the statement did not work. Students could apply a variety of concepts and properties related to the problem solving. The concepts of inequality, factoring, set and logic were used to determine the solution set of inequality problem. Students constructed knowledge by connecting between what is being faced with the existing knowledge. Students did not memorize the concepts, rules, procedures, or properties but they understood it well by relating to their knowledge previously. The learning process students' $R_f$PR is consistent with the meaningful learning theory [22]. According to the terminology of Hiebert and Lefevre [23], students' knowledge is categorized by conceptual knowledge. Meanwhile, if it is viewed by the terminology of understanding, the students have a relational understanding [24] or conceptual understanding [25].

## V. Conclusion

In this study students performed plausible reasoning well in the problem solving. Students also could overcome the difficulty during the problem solving because of applying reflective thinking process maximally. Therefore, in the learning process the educators should provide greater opportunities for students to take reflection process so that they can find the solution of the problem and perform reflective plausible reasoning optimally. Another result of this study is a few students performed plausible reasoning during the inequality problem solving. Most students used the learning experience previously without deep understanding. In other words, many students performed EE. This can also be found in previous studies (e.g., [7], [9], [10]), which reveal that EE is more dominant than PR. Moreover, many students applied superficial reasoning. Therefore, it is very essential for an educator to familiarize students to use plausible reasoning by explaining the process of solving the problem, justifying the problem solving the steps, and convincing the truth of the result. Further research is required to examine the students' failure in plausible reflective reasoning. In addition, there is still an open study to investigate the trigger of students doing EE.

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